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SOLVING LATERAL-TORSIONAL BUCKLING PROBLEMS IN THIN-WALLED BISYMMETRIC BEAM USING STODOLA-VIANELLO ITERATION METHOD

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Abstract

Thin-walled beams are susceptible to lateral torsional buckling (LTB) and can fail by LTB even when their material strengths have not been attained. The safe designs of thin-walled beams thus require LTB analysis to determine minimum LTB load. This paper aims to develop LTB load solutions of thin-walled bisymmetric beams using Stodola-Vianello iteration method (SVIM). The equations of LTB are differential equations derived from equilibrium conditions, incorporating bending, torsion, and warping effects. These equations are then transformed to more amenable iteration equations using SVIM. The SVIM uses successive integrations of the LTB equations to derive a system of iteration equations which is expressed for the arbitrary buckling mode, n. This work illustrates the use of the developed iteration equations to derive LTB solutions for simply supported thin-walled beams under constant end moment. Exact sinusoidal shape functions for the nth buckling mode derived from the governing equation for buckling are used for the two buckling displacement functions u(x), $\phi(x)$ to derive the SVIM equations for the subsequent (n + 1)th iteration. Convergence at nth iteration is used for finding the stability equation that is solved for the eigenvalue which is used to find the buckling load. The method yielded closed form LTB solutions that were identical with previous solutions obtained using Ritz methods, finite Fourier sine transform method, least square weighted residual method and classical Navier series method.

1.0 INTRODUCTION

When a beam that is subjected to loads undergoes unrestrained, excessive simultaneous lateral displacement and twisting at the imminence of failure, it is said to undergo lateral-torsional buckling (LTB). LTB is a common failure feature of unrestrained steel beam. For such beams, top flange can freely displace laterally and simultaneously rotate. [1], [2], [3], [4]. The determination of the minimum load that causes LTB is an important step in the design of beams prone to such failures. The significance of LTB is demonstrated by the vast quantum of research interest in LTB analysis. Several research works on elastic LTB analysis of beams have been done using equilibrium or total energy minimization techniques Jiki [5], Fu [6], Ike [7].

Timoshenko and Gere [8] pioneered research on the development of analytical solutions for minimum buckling loads of bisymmetric beams on fork supports. Their work assumed that torsion and lateral

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deformations are completely prevented but out-of-plane rotations and cross-sectional warping are not restrained. Badriri and Papp [9] presented a review of methods of LTB for various ends supports and for various loads on beams. Jiki [5]] used and energy method to derive stability matrices for LTB of beams and found minimum LTB loads of simply supported beams carrying constant moments at both supports but with uniform torsion neglected. The omission of torsion is the defect of Jiki's solution.

Kumar [10] presented a review of Rayleigh-Ritz technique for LTB solutions. Attard [11] used finite element method for the LTB of beams. Sahraei [12] studied LTB of beams using matrix based methods and obtained accurate buckling load solutions. Oguaghamba et al [13] explored Ritz variational methodology (RVM) for deriving analytical instability loads for bisymmetric thin-walled beams. Ike [7] presented energy formulation of buckling of thin-walled column. Euler-Lagrange equation was applied to obtain stability equation. Ike et al [14] used Fourier cosine series methodology (FCSM) for obtaining accurate analytical solutions to general buckling load problems. In another work, Ike et al [15] used a modification of single Fourier cosine integral transformation methodology for solving critical instability loads in shear deformable beams. Ike et al [16] have explored Laplace transformation methodology (LTM) in stability studies of beams. The LTM transformed the equation to an algebraic equation, and took automatic consideration of the initial conditions. Ike et al [17] utilized weighted residual method in LTB.

Oguaghamba et al [18] utilized Finite Fourier integral transform methodology for stability solutions of bisymmetric beams under simple supports. Onah et al [19] derived closed-form solutions for stability of thick beams. They solved the governing equation in closed-form using the method of trial functions.

Szychowski [20] studied the stability of open cross-section bars under cross-sectional warping torsion. Argyridi and Sapountzakis [21] studied LTB of composite beam.

Nguyen et al [22] studied LTB in beam under both end moments and transverse loads. Zhang et al [23] used linear stability theory to derive LTB solutions of I-beams under linearly distributed moments. Juliusz [24] studied LTB in tapered steel beam. Abdul Hamed [25] studied LTB in monosymmetric beam. Yilmaz and Kirac [26] studied LTB analysis of singly loads of monosymmetric beams.

Soltani and Asgarian [27] investigated the LTB load analysis of thin-walled beams having monosymmetric sections using ordinary finite difference methodology (FDM) and obtained reasonably accurate buckling solutions. The accuracy of the critical buckling load solutions increase with mesh refinement but increased computerized effort was needed.

Lebastard et al [28] studied LTB for of beams warping restrained at supports. They derived exact solutions by using infinite power series method of representation of twist rotation to solve the differential equations of equilibrium. Piotrowski and Szydrowski [29] studied LTB of beams with elastic restraints at the supports.

Ozbasaram et al [30] explored the principle of potential energy minimization to study LTB analysis for bisymmetric cantilever I beams and obtained accurate solutions. Sercer and Uzun [31] investigated elastic LTB for beams carrying point load and linear moment gradient. Chen et al [32] derived critical LTB loads of beams under various transverse loads.

A review of literature shows that though several analytical and numerical methods have been used for LTB of thin-walled beams, SVIM has not been utilised. The demonstrated effectiveness including the accuracy of SVIM in solving of buckling of beams rested on elastic soil of the one-parameter and two-parameter types suggest that the method could be explored for thin-walled buckling analysis. This papers' aim is to explore the SVIM for lateral stability load solutions of beams with bisymmetric cross-sections, and to specifically present SVIM LTB bisymmetric beam under constant end moments.

The novelty of the work is that this is the first attempt at exploring applications of SVIM to buckling analysis of beams with thin-walls. A merit of the SVIM is its simplicity, and ease of use because it converts the stability equations to a set of easier to solve algebraic iteration equations.

2.0 METHODOLOGY

2.1 Equations of Stability

The equations of LTB of homogeneous prismatic bi-symmetric beams are the system of differential equations:[1].

$$EI_z u^{iv} - M\phi'' = 0 \tag{1}$$

$$EI_w \phi^{iv} - GJ\phi'' - Mu'' = 0 \tag{2}$$

Where, E denotes Young's modulus, G denotes shear modulus I_z denotes moment of inertia about minor cross-sectional axis. J is the Saint Venant torsional parameter, I_w denotes warping parameter, u denotes lateral displacement of the shear center, ϕ denotes the rotational displacement of shear center, or twist rotation angle. Equations (1) and (2) are the LTB governing equations, where critical buckling occurs when the determinant of the system vanishes, leading to an eigenvalue problem.

2.2 Stodola-Vianello Iteration Formula for the Equations of Stability

SVIM uses successive integrations of the domain equations to re-express them as iteration equations, as illustrated by application of the SVIM to beams on elastic foundations problem by Ike [33, 34, 35]. Hence, by successive integration of Equation (1).

$$u'''(x) - \frac{M}{EI_z}\phi'(x) + c_1 = 0 \tag{3}$$

Where, c_1 is a constant of integration.

Integrating again,

$$u''(x) - \frac{M}{EI_z}\phi(x) + c_1 x + c_2 = 0 \tag{4}$$

c_2 is another constant of integration for the second successive integration.

Integrating again,

$$u'(x) - \frac{1}{EI_z}\int_0^x \phi(x)\,dx + \frac{c_1 x^2}{2} + c_2 x + c_3 = 0 \tag{5}$$

c_3 is the third successive integration constant.

Integrating again,

$$u(x) - \frac{1}{EI_z}\int_0^x\int_0^x M\phi(x)\,dx\,dx + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4 = 0 \tag{6}$$

c_4 is the fourth successive integration constant.

The corresponding SVIM iteration equations for the nth iteration are:

$$u'''_{n+1}(x) = \frac{M}{EI_z}\phi'_n(x) - c_1 \tag{7}$$

$$u''_{n+1}(x) = \frac{M}{EI_z}\phi_n(x) - (c_1 x + c_2) \tag{8}$$

$$u'_{n+1}(x) = \frac{1}{EI_z}\int_0^x M\phi_n(x)\,dx - \left(\frac{c_1 x^2}{2} + c_2 x + c_3\right) \tag{9}$$

$$u_{n+1}(x) = \frac{1}{EI_z}\int_0^x\int_0^x M\phi_n(x)\,dxdx - \left(\frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4\right) \tag{10}$$

A similar implementation of successive integrations to Equation (2) would yield as follows:

$$\phi'''(x) - \frac{GJ}{EI_w}\phi'(x) - \frac{M}{EI_w}u'(x) + \bar{c}_1 = 0 \tag{11}$$

$\bar{c}_1$ is the integration constant.

$$\phi''(x) - \frac{GJ}{EI_w}\phi(x) - \frac{Mu(x)}{EI_w} + \bar{c}_1 x + \bar{c}_2 = 0 \tag{12}$$

$$\phi'(x) - \frac{GJ}{EI_w}\int_0^x \phi(x)\,dx - \frac{1}{EI_w}\int_0^x Mu(x)\,dx + \frac{\bar{c}_2 x^2}{2} + \bar{c}_2 x + \bar{c}_3 = 0 \tag{13}$$

$$\phi(x) - \frac{GJ}{EI_w}\int_0^x\int_0^x \phi(x)\,dxdx - \frac{1}{EI_w}\int_0^x\int_0^x Mu(x)\,dxdx + \frac{\bar{c}_1 x^3}{6} + \frac{\bar{c}_2 x^2}{2} + \bar{c}_3 x + \bar{c}_4 = 0 \tag{14}$$

The iteration equations are:

$$\phi'''_{n+1}(x) = \frac{GJ}{EI_w}\phi'_n(x) + \frac{Mu'_n(x)}{EI_w} - \bar{c}_1 \tag{15}$$

$$\phi''_{n+1}(x) = \frac{GJ}{EI_w}\phi_n(x) + \frac{Mu_n(x)}{EI_w} - (\bar{c}_1 x + \bar{c}_2) \tag{16}$$

$$\phi'_{n+1}(x) = \frac{GJ}{EI_w}\int_0^x \phi_n(x)\,dx + \frac{1}{EI_w}\int_0^x Mu_n(x)\,dx - \left(\frac{\bar{c}_2 x^2}{2} + \bar{c}_2 x + \bar{c}_3\right) \tag{17}$$

$$\phi_{n+1}(x) = \frac{GJ}{EI_w}\int_0^x\int_0^x \phi_n(x)\,dxdx + \frac{1}{EI_w}\int_0^x\int_0^x Mu_n(x)\,dxdx - \left(\frac{\bar{c}_1 x^3}{6} + \frac{\bar{c}_2 x^2}{2} + \bar{c}_3 x + \bar{c}_4\right) \tag{18}$$

2.3 SVIM for the Buckling of Bisymmetric Beams under equal Constant End Moment M_0

The simply supported bisymmetric beam buckling under constant bending moments M_0 at the simple supports as illustrated in Figure 1 is investigated.

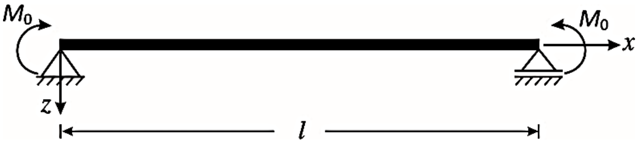

Figure 1: Simply supported bisymmetric beam under constant bending moments M_0 at both supports.

The boundary conditions (BCs) are:

$$\phi(0) = \phi(l) = 0, \quad \phi''(0) = \phi''(l) = 0$$

$$u(0) = u(l) = 0, \quad u''(0) = u''(l) = 0 \tag{19}$$

The functions that satisfy the BCs at the nth buckling mode are:

$$u(x) = u_n \sin\frac{n\pi x}{l} \tag{20}$$

$$\phi(x) = \phi_n \sin\frac{n\pi x}{l} \tag{21}$$

The SVIM equations then become:

$$u''_{n+1}(x) = \frac{M_0}{EI_z}\phi_n \sin\frac{n\pi x}{l} - (c_1 x + c_2) \tag{22}$$

$$u_{n+1}(x) = \frac{1}{EI_z}\int_0^x\int_0^x M_0\,\phi_n \sin\frac{n\pi x}{l}\,dxdx - \left(\frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4\right) \tag{23}$$

$$\phi''_{n+1}(x) = \frac{GJ}{EI_w}\phi_n \sin\frac{n\pi x}{l} + \frac{M_0 u_n}{EI_w}\sin\frac{n\pi x}{l} - (\bar{c}_1 x + \bar{c}_2) \tag{24}$$

$$\phi_{n+1}(x) = \frac{GJ}{EI_w}\int_0^x\int_0^x \phi_n \sin\frac{n\pi x}{l}\,dxdx + \frac{1}{EI_w}\int_0^x\int_0^x M_0 u_n \sin\frac{n\pi x}{l}\,dxdx - \left(\frac{\bar{c}_1 x^3}{6} + \frac{\bar{c}_2 x^2}{2} + \bar{c}_3 x + \bar{c}_4\right)$$

…(25)

$$\int_0^x\int_0^x \phi_n \sin\frac{n\pi x}{l}\,dxdx = -\left(\frac{l}{n\pi}\right)^2 \phi_n \sin\frac{n\pi x}{l} \tag{26}$$

$$\int_0^x\int_0^x u_n \sin\frac{n\pi x}{l}\,dxdx = -\left(\frac{l}{n\pi}\right)^2 u_n \sin\frac{n\pi x}{l} \tag{27}$$

Then Equations (23) and (25) simplify to:

$$u_{n+1}(x) = \frac{-M_0}{EI_z}\left(\frac{l}{n\pi}\right)^2 \phi_n \sin\frac{n\pi x}{l} - \left(\frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4\right) \tag{28}$$

$$\phi_{n+1}(x) = \frac{-GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2 \phi_n \sin\frac{n\pi x}{l} - \frac{M_0}{EI_w}\left(\frac{l}{n\pi}\right)^2 u_n \sin\frac{n\pi x}{l} - \left(\frac{\bar{c}_1x^3}{6} + \frac{\bar{c}_2x^2}{2} + \bar{c}_3x + \bar{c}_4\right) \tag{29}$$

Enforcing BCs yield:

$$u''_{n+1}(0) = -c_2 = 0 \tag{30}$$

$$\phi''_{n+1}(0) = -\bar{c}_2 = 0 \tag{31}$$

$$u''_{n+1}(l) = \frac{M_0\phi_n}{EI_z}\sin n\pi - c_1 l = 0 \tag{32}$$

$$\therefore\ c_1 = 0 \tag{33}$$

$$\phi''_{n+1}(l) = \frac{GJ}{EI_w}\phi_n \sin n\pi + \frac{M_0u_n}{EI_w}\sin n\pi - \bar{c}_1 l = 0 \tag{34}$$

$$\therefore\ \bar{c}_1 = 0 \tag{35}$$

$$u_{n+1}(0) = -c_4 = 0 \tag{36}$$

$$\phi_{n+1}(0) = -\bar{c}_4 = 0 \tag{37}$$

$$u_{n+1}(l) = -\frac{M_0}{EI_z}\left(\frac{l}{n\pi}\right)^2 \phi_n \sin n\pi - c_3 l = 0 \tag{38}$$

$$\therefore\ c_3 = 0 \tag{39}$$

$$\phi_{n+1}(l) = -\frac{GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2 \phi_n \sin n\pi - \frac{M_0}{EI_w}\left(\frac{l}{n\pi}\right)^2 u_n \sin n\pi - \bar{c}_3 l = 0 \tag{40}$$

$$\therefore\ \bar{c}_3 = 0 \tag{41}$$

3.0 RESULTS AND DISCUSSION

3.1 Results

The SVIM equations then become:

$$u_{n+1}(x) = -\frac{M_0}{EI_z}\left(\frac{l}{n\pi}\right)^2 \phi_n \sin\left(\frac{n\pi x}{l}\right) \tag{42}$$

$$\phi_{n+1}(x) = -\frac{GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2 \phi_n \sin\left(\frac{n\pi x}{l}\right) - \frac{M_0}{EI_w}\left(\frac{l}{n\pi}\right)^2 u_n \sin\left(\frac{n\pi x}{l}\right) \tag{43}$$

At convergence of the SVIM,

$$u_{n+1}(x) = u_n(x) \tag{44}$$

$$\phi_{n+1}(x) = \phi_n(x) \tag{45}$$

The SVIM equations then become:

$$u_n \sin\frac{n\pi x}{l} = -\frac{M_0}{EI_z}\left(\frac{l}{n\pi}\right)^2 \phi_n \sin\frac{n\pi x}{l} \tag{46}$$

$$\phi_n(x)\sin\frac{n\pi x}{l} = -\frac{GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2 \phi_n \sin\frac{n\pi x}{l} - \frac{M_0}{EI_w}\left(\frac{l}{n\pi}\right)^2 u_n \sin\frac{n\pi x}{l} \tag{47}$$

or,

$$\left(u_n + \frac{M_0}{EI_z}\left(\frac{l}{n\pi}\right)^2 \phi_n\right)\sin\frac{n\pi x}{l} = 0 \tag{48}$$

$$\left\{\phi_n + \frac{GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2 \phi_n + \frac{M_0}{EI_w}\left(\frac{l}{n\pi}\right)^2 u_n\right\}\sin\frac{n\pi x}{l} = 0 \tag{49}$$

Thus,

$$u_n + \frac{M_0}{EI_z}\left(\frac{l}{n\pi}\right)^2 \phi_n = 0 \tag{50}$$

$$\frac{M_0}{EI_w}\left(\frac{l}{n\pi}\right)^2 u_n + \left(1 + \frac{GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2\right)\phi_n = 0 \tag{51}$$

In matrix form, this gives:

$$\begin{pmatrix} 1 & \frac{M_0}{EI_z}\left(\frac{l}{n\pi}\right)^2 \\ \frac{M_0}{EI_w}\left(\frac{l}{n\pi}\right)^2 & \left(1 + \frac{GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2\right) \end{pmatrix}\begin{pmatrix} u_n \\ \phi_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{52}$$

For nontrivial solutions, $\begin{pmatrix} u_n \\ \phi_n \end{pmatrix} \neq 0$ the characteristic buckling equation is:

$$\begin{vmatrix} 1 & \frac{M_0}{EI_z}\left(\frac{l}{n\pi}\right)^2 \\ \frac{M_0}{EI_w}\left(\frac{l}{n\pi}\right)^2 & \left(1 + \frac{GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2\right) \end{vmatrix} = 0 \tag{53}$$

Expanding the determinant gives:

$$\left(1 + \frac{GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2\right) - \frac{M_0}{EI_z}\left(\frac{l}{n\pi}\right)^2 \frac{M_0}{EI_w}\left(\frac{l}{n\pi}\right)^2 = 0 \tag{54}$$

Expanding,

$$\left(1 + \frac{GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2\right) - \frac{M_0^2}{EI_wEI_z}\left(\frac{l}{n\pi}\right)^4 = 0 \tag{55}$$

Solving for M_0 gives:

$$M_0^2 = EI_wEI_z\left(\frac{n\pi}{l}\right)^4\left(\frac{GJ}{EI_w}\left(\frac{l}{n\pi}\right)^2 + 1\right) \tag{56}$$

Simplifying,

$$M_0^2 = EI_wEI_z\left(\frac{GJ}{EI_w}\left(\frac{n\pi}{l}\right)^2 + \left(\frac{n\pi}{l}\right)^4\right) \tag{57}$$

Alternatively,

$$M_0^2 = EI_wEI_z\left(\frac{n\pi}{l}\right)^2\left(\frac{GJ}{EI_w} + \left(\frac{n\pi}{l}\right)^2\right) \tag{58}$$

Hence,

$$M_0^2 = EI_z\left(\frac{n\pi}{l}\right)^2\left(GJ + EI_w\left(\frac{n\pi}{l}\right)^2\right) \tag{59}$$

Taking the square root, the buckling load at the *n*th mode becomes:

$$M_0(n) = \frac{n\pi}{l}\sqrt{EI_z\left(GJ + EI_w\left(\frac{n\pi}{l}\right)^2\right)} \tag{60}$$

The critical buckling moment M_{cr} is the least M_0 which occurs when $n = 1$. So,

$$M_{cr} = M_0(n = 1) = \frac{\pi}{l}\sqrt{EI_z\left(GJ + EI_w\left(\frac{\pi}{l}\right)^2\right)} \tag{61}$$

This is identical with M_{cr} obtained by Hama (1), Timoshenko and Gere [8], Oguaghamba et al [13, 18], Sahraei [12].

This is alternatively expressed as:

$$M_{cr} = \frac{\pi}{l}\sqrt{EI_zGJ + EI_zEI_w\left(\frac{\pi}{l}\right)^2} \tag{62}$$

$$M_{cr} = \frac{\pi}{l}\sqrt{EI_zGJ\left(1+\frac{EI_w}{GJ}\frac{\pi^2}{l^2}\right)} \tag{63}$$

$$M_{cr} = \frac{\pi}{l}\sqrt{EI_zGJ}\sqrt{1+\frac{EI_w}{GJ}\frac{\pi^2}{l^2}} \tag{64}$$

$$M_{cr} = k_b\frac{\sqrt{EI_zGJ}}{l} \tag{65}$$

$$k_b = \pi\sqrt{\left(1+\frac{EI_w}{GJ}\frac{\pi^2}{l^2}\right)} \tag{66}$$

The values of k_b are computed for certain values of $\frac{GJl^2}{EI_w}$, and displayed in Table 1 together with previously obtained solutions by Oguaghamba et al [13] and Timoshenko and Gere [8]

Table 1: Comparison of variation of Critical Buckling Load Factors, k_b with $GJl^2/(EI_w)$

$\frac{GJl^2}{EI_w}$	**Present results**	**Timoshenko and Gere [8]**	**Oguaghamba et al [13]**
0	∞	∞	∞
0.1	31.3681	31.3681	31.3681
1.0	10.3575	10.3575	10.3575
2	7.6534	7.6534	7.6534
4	5.8499	5.8499	5.8499
6	5.1093	5.1093	5.1093
8	4.6953	4.6953	4.6953
10	4.4284	4.4284	4.4284
12	4.2411	4.2411	4.2411
16	3.9947	3.9947	3.9947
20	3.8393	3.8393	3.8393
24	3.7321	3.7321	3.7321
28	3.6536	3.6536	3.6536
32	3.5936	3.5936	3.5936
36	3.5462	3.5462	3.5462
40	3.5078	3.5078	3.5078
100	3.2930	3.2930	3.2930
∞	π	π	π

When $I_w = 0$,

$$M_{0\,cr} = \frac{\pi}{l}\sqrt{(EI_zGJ)} \tag{67}$$

Equation (67) is identical with results by Attard [11], Jiki [5] obtained $M_{0\,cr}$, using Rayleigh-Ritz method as:

$$M_{0\,cr} = \frac{3.46}{l}\sqrt{(EI_zGJ)} \tag{68}$$

3.2 Discussion

This study has presented SVIM for the closed form LTB solutions of thin-walled beam under constant end moments. The governing differential equation (GDE) is a system of two equations in terms of two unknown displacement functions, $u(x)$ and $\phi(x)$. For beam under transverse load distribution the GDE results in a variable parameter differential equation that is difficult to solve in closed form. For the case of thin-walled beam under constant end moments, the SVIM used with the sinusoidal buckling functions for $u(x)$ and $\phi(x)$ converts the BVP to system of iterative equations with integration constants that are determined using BCs. The convergence criteria which requires that the nth and $(n + 1)$th iteration for $u(x)$ and $\phi(x)$ become equal are used to derive the buckling equation in matrix form as Equation (52). The condition for nontrivial solutions yield the buckling equation for nth buckling mode in closed form as Equation (60). The critical LTB load is associated with the first buckling mode and is found in closed form as Equation (61) which is identical with previous closed form critical LTB solutions by [1], [8], [12], [13] and [18]. The critical LTB solutions are tabulated using critical buckling factors k_b shown in Table 1.

Table 1 shows that the critical LTB load factor obtained in the present work using SVIM is identical with the previous solutions of [8] and [13]. The critical buckling load factor k_b depends on the torsional and warping parameters $\frac{GJl^2}{EI_w}$. When $\frac{GJl^2}{EI_w} = 0$, k_b is infinite and for large values of $\frac{GJl^2}{EI_w}$, k_b approaches π.

4.0 CONCLUSION

The SVIM was utilised to determine analytical LTB loads of thin-walled beams with bisymmetrical cross-sections. The SVIM converts the GDE of LTB to a system of iterative equations; which is simplified by finding the integration constants using boundary conditions.

In conclusion:

(i) Exact sinusoidal buckling shape functions for $u(x)$ and $\phi(x)$ in the SVIM gave exact LTB load solutions for thin-walled bisymmetric beams under constant end moments.

(ii) The convergence of the buckling displacements at the nth iteration was used to construct the buckling equation of the problem in matrix form.

(iii) The determinant of the coefficient matrix is equated to zero in finding the buckling equation.
(iv) Eigenvalues are the roots of the buckling equation.
(v) Critical LTB load corresponded with the first mode (n=1), and are the same as previously found LTB loads obtained via Ritz variational method by Oguaghamba et al [13], Navier series method by Timoshenko and Gere [8], finite Fourier sine transform method by Oguaghamba et al [18] and least squares weighted residual method by Ike et al [17].
(vi) The critical buckling load depends upon the elastic and geometric properties of the bisymmetric beam and can be determined from the dimensionless buckling factor k_b which depends on $GJl^2/(EI_w)$. Then M_{cr}, the critical buckling moment is found using Equation (65).

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