MODELLING OF FAR-FIELD MIXING OF INDUSTRIAL EFFLUENT PLUME IN
AMBIENT RECEIVING WATER: THE IKPOBA RIVER HUB EXAMPLE

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ABSTRACT
A study was initiated with a view to understanding the mathematical physics of industrial effluent plume dispersion in Ikpoba River which has resulted in the development of a sacrificial stretch along the river course and which made water body from that stretch unfit for various uses on account of resulting water quality parameters that are above threshold values. This study sought to describe the dynamics of advective and dispersive transport of the effluent plume in the river and also ascertain the extent of its effect from discharge location to downstream far-field region. A homogenous differential equation was used as analytics to describe the physical process that describes the wave of dispersion in water as a Doppler shift run. The model developed provides a geometric illustration of the dispersion process which gives a graphic detail of wavefronts distribution of effluent matter. Another important result of this study is that in the initially 3-D mixing process, the drop-off of the maximum mass concentration is relatively fast but thereafter the concentration gradually loses intensity downstream and at 1800m the concentration variations become below the water quality threshold criteria. In effect, industrial effluent loading is continuous and concentrated thereby causing a stretch of 1.8km sacrificial region to develop along the river course.

Keywords: Far-field Mixing, Sacrificial Stretch, Doppler Shift run, Complete Vertical Mixing, Pollution Concentration Parameters

1. INTRODUCTION
The mathematical physics of dispersion of industrial effluent plumes emanating from conduit outfalls into Ikpoba River is yet to be fully understood. Apart from the seminal study [1] that investigated the contaminant concentration distribution downstream of the river, no other study to our knowledge has provided the mathematical behaviour of dispersion of plume in the river in question.

A good understanding of the mechanism of pollutant dispersion in the river may offer the following advantages:
(i) it will provide insight into the nature and scale of potential hazards the effluent discharge pose in the ambient receiving water[2];
(ii) the solution technique to the water pollution problem can be easily proffered[2]; and
(iii) accurate evaluation of the hydraulic conditions (speed, diffusion, advection, turbulence, transport and transformation, etc) of the polluted portion of the river aids in the determination of river water quality[3].

Although many studies have addressed the biochemical analysis of the polluted water [4-9], the dynamics of molecular diffusion and advection in the river has been rarely investigated. The current study presents a mathematical analysis on the strictly physical processes that affect pollutants solute transport which plays a vital role in the determination of the fate of the effluent concentration in the water body.

Also, deserving of note is the fact that the development of the theory of dilution of effluents in water body is credited to the seminal work by [10]. Further, the study [11] provided an extension of the theory by considering two dimensional advection-dispersion equation. Moreover, a general review of modelling pollution dispersion of water bodies by [12] is a comprehensive treatment of the extant researches on modelling situations and provides a concise summary of current capabilities in coastal pollution, ecosystem and water quality modelling.

In particular the paper highlighted in part modelling in which there are: diffusion and dispersion, lateral dispersion, quasi-two-dimensional turbulence, Lagrangian chaos, and numerical diffusion. The reviewer noted conclusively that considerable advances had been made in recent years in the
elements needed for predicting coastal water quality operationally, namely three-dimensional hydrodynamic shelf-sea models and ecological, pollutant and sediment models. The author further observed however that more still need to be done to improve on the existing models and that research remains very active in each of the individual areas he covered.

At local level, the study which was undertaken to ascertain the trace metal profile in fish, sediments and water of Ikpoba river analysed the trace metal concentration in the river and concluded that there is a relatively high level of Iron (Fe) and Lead (Pb) which exceeded the levels recommended by both the World Health Organization (WHO) and Food and Agricultural Organization (FAO) of the United Nations. Additionally, the studies: [13-18] dwelt on the biochemical analysis of various pollutants in Ikpoba River but provided little insight on the advection / dispersion mechanism of these pollutants and their mixing in the river downstream of the outfall.

Moreover, the study by [19] carried out economic analysis and modelling of water quality in relation to its sustainable assimilative capacity. The study focused on environmental policy designs targeted at holding long term emissions to certain tolerable limits that may otherwise be termed dangerous in which case the policy might prescribe outright ban to such fading substance if substitute product(s) exist. The study, advocates for the use of clean technology but cautioned that the economic and ecological consequences such as welfare loss of outright ban on polluting technology need to be considered also. Again, paper [20] argues that there is a minimum instream flow requirements for survival of fish and other aquatic creatures to survive in a body of water noting that augmentation of river flow boosts the Dissolved Oxygen (DO) level. The study indicated that contamination generated from upstream development and human activities introduces a significant amount of nutrients into large natural stream of water thereby fostering the development of hypoxia or eutrophication process that spoils public water resources and requiring costly remediation to restore optimal DO level. The study also presents the development of hydrodynamic and water quality model that addresses such issues. Relatedly, [21] carried out a similar study to model the shadow price of assimilative capacity in optimal flow pollution control and concluded by drawing attention to the need to include environmental regenerative conditions in economic frameworks.

From the above review, it is evident that some work had been done on the effluent loading into Ikpoba River and the resultant degradation of its water quality but environmental fluid mechanics theoretical support and the theory of mixing of the pollutants in the near field and the far field are currently lacking. The aim of this paper therefore is to develop a mathematical model that describes the concentration distributions and transport of contaminants in Ikpoba River so as to fully ascertain the extent of sacrificial stretch and hence the safe distance at which pollutants’ concentrations can be truly regarded as being below certain threshold criteria. This is significant in view of the fact that engineering practice is intended to use the knowledge of natural transport and transformation process in water bodies to design projects that minimize the probability of occurrence of toxic concentrations while maintaining an affordable budget. Further, [3] corroborates this notion by stressing that the specification of where in water body the environmental quality standards apply is vital.

2. MATERIALS AND METHODS

Previous studies have established the fact that swift river flow is strongly linked to effective effluent dispersion and hence assimilative capacity; see for example, [3, 22]. Arising from this premise therefore, we used Swoffer Meter to determine the speed of Ikpoba River around the effluent discharge point-3($P_3$) located near a large brewery. From the readings the shear velocity was calculated. A previous study, [23] showed that $P_3$ had outstanding values of water quality parameter vector, see Figure 1 and Table 1. In the study, river water samples for physiochemical, microbiological and heavy metal analysis were collected from four predetermined points. At each point, upstream and downstream measurements of pollutants’ parameters were made. Sample specimens of the river water were collected using 50ml reagent bottles. The oxygen fixation for the specimens was done with Winkler’s solution A and B. The samples meant for heavy metal analysis were collected with 1 litre plastic containers. Distilled-water diluted-Nitric acid was required to maintain the oxidation state of the elements and also to prevent metals from adhering to the walls of the container. The temperature of samples was measured with mercury-in-glass thermometer (0-100°C). The samples were preserved with ice chest and refrigerator before chemical analysis.
\[ W = Q_e C_e \]

where \( W \) is the mass loading rate \([M/T]\), \( Q_e \) is the volumetric flow rate in \( m^3 s^{-1} = 100 \) litres/sec and \( C_e \) is the solute concentration mg/litre (which varies according to pollutant parameter).

The drop off of maximum mass concentration is given by:

\[ C_{\text{max}} = \frac{1}{x}[3, 22] \]

In practice,

\[ C_{\text{max}} = x^2[3, 22] \]  

By intuitive knowledge, any point on the locus of wave of dispersion is equidistant from the orifice within the orifice region and 3-D mixing zone.

\[ \frac{dy}{dx} = \frac{y-y'}{x-x'} \]

or \((x-x')y' = Y - y\)

At \( T, Y = 0 \) so that \( X = OT = \frac{y}{y'} + x\)

\[ MT = \sqrt{y^2 + \left(\frac{y}{y'}\right)^2} \]

Since \( OT^2 = MT^2 \) then \( \left(x - \frac{y}{y'}\right)^2 = y^2 + \left(\frac{y}{y'}\right)^2 \)

\[ \frac{dy}{dx} = \frac{2x}{x^2 - y^2} \]

Setting \( y = Ux \), then

\[ \frac{dy}{dx} = U + x \frac{dx}{dx} \]

\[ \therefore U + x \frac{dy}{dx} = \frac{2xUx}{x^2 - U^2 x^2} \]

which gives

\[ \text{Table 1: Vector of water quality parameters at station 3(P3)} \]

<table>
<thead>
<tr>
<th>Parameters</th>
<th>PO₄</th>
<th>NO₃</th>
<th>Cd</th>
<th>Cu</th>
<th>Fe</th>
<th>Pb</th>
<th>Turb.</th>
<th>Temp.</th>
<th>Fcl col</th>
<th>pH</th>
<th>DO</th>
<th>BOD</th>
<th>COD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Station 3</td>
<td>0.868</td>
<td>0.933</td>
<td>0.77</td>
<td>0.0503</td>
<td>5.062</td>
<td>0.717</td>
<td>30</td>
<td>27.433</td>
<td>126.33</td>
<td>6.16</td>
<td>0.267</td>
<td>16.467</td>
<td>277.633</td>
</tr>
</tbody>
</table>

\[ \text{Fig. 1: The Study Area (A) Map of Nigeria Showing Benin City in Edo State, (B) Map of Ikpoba River Showing Four Sampling Stations} \]
\[
\log_2(U + U^3) - 2 \log(1 + U^2) = \log x + C
\]
\[
\frac{U + U^3}{(1 + U^2)^2} = xC \quad \text{or} \quad \frac{y}{x} + \frac{y^3}{x^2} = xC
\]
\[
y = Cx^2 \left(1 + \frac{y^2}{x^2}\right)
\]
Which simplifies to \(y = \frac{C}{x^2} \left(2x^2 + y^2\right)\)

And by setting \(\frac{1}{2A} = C\), we obtain:

\[x^2 + (y - A)^2 = A^2\]  (4)

As \(A\) varies, we have a family of curves depicted in Figure 4.

By implication, the distance of the tangent at \(M\) to the fiducial (longitudinal) line downstream of river \(x\) is equal to distance from origin (orifice) to the intercept of the tangent to the fiducial line. However, the mathematical physics representation of Figure 4 is illustrated in Figure 5 with the axes swapped.

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**Figure 2: Schematic Sketch of Mixing Zones in Ikpoba River**

**Figure 3: Geometry of Effluent Plume Propagation**
4. RESULTS AND DISCUSSION

4.1 Result of Application of Homogenous Differential Equation

The model developed is of the form

\[ x^2 + (y - A)^2 = A^2 \]  

which is a circle with x-axis fixed at the origin while y-axis is steadily shifting at the speed of the river, i.e. longitudinally to create a doppler shift run, see Figures 4 and 5. The model illustrates the dispersion process that portrays lurid depiction of wavefronts distribution of effluent matter. As a matter of fact, at this juncture, three hydrodynamic features namely – the initial jet characteristics of momentum flux, buoyancy flux due to density differences and outfall geometry- influence the effluent trajectory and degree of mixing. The result of this plume propagation in Ikpoba River corroborates with the concentration dispersion illustrated in [2].

It should however be observed that the dispersion of effluent in this circular profile is actually being effected in spherical form i.e in three dimensions (3-D): vertically down towards the river bed, laterally towards the river banks, and longitudinally along the river course, downstream. Our model is able to describe this phenomenon in two dimensions at a time.

4.2 Determination of mass concentration distribution in 3-D mixing zone

We employ the relation

\[ C_x = \frac{1}{x} \]

For contaminant vector
\[ C_x = [PO_4, NO_3, Cd, \ldots \text{COD}] = [0.868, 0.933, 0.503, \ldots, 277.633] \]

For instance, \[ C_{5(PO_4)} = [0.868 \times \frac{1}{5}] = 0.174 \]

And

\[ C_{9(COD)} = [277.633 \times \frac{1}{9}] = 30.848 \]

Table 2 shows the dispersion of mass concentration of the effluent as plume from the orifice. Row 1 of the table represents a vector of contaminants taken as sample from the discharge point using the relation: \( C_x = \frac{1}{x} \) where \( x \) denotes the longitudinal distance measured from the orifice, the fading effluent concentration downstream up to 10 metres, within the 3-D mixing zone is illustrated. The order of sequence of the contaminants is as determined by eigenvalue analytics used in earlier study by the authors [23]. See Table 2 for complete results of computation.

### 4.3 Determination of location of complete vertical mixing

For the determination of the location of complete vertical mixing of the plume \( (L_{mv}) \) we employed

\[ L_{mv} = 0.4 \frac{u_h^2}{E_z} \]  

(5)

Where \( U \) is the average velocity, \( h \) is the depth of the river and \( E_z = \alpha_z U h \) is the vertical eddy diffusivity

From [28],

\[ \alpha_z = 0.07 \pm 50 \]

From [3], frictional velocity \( U^* \) is given by:

\[ U_1 = (0.05 \text{ to } 0.10)U \]

From our measurements, \( U = 1.35 \text{m/s, } h = 4 \text{m} \)

\( B = \) River width = 30 metres

\( E_z = \alpha_z U h \)

\( E_z = 0.07x(0.1x1.35)x4 = 0.0378 \)

\[ \therefore L_{mv} = \frac{0.4 \times 1.35 \times 4^2}{0.0378} = 229 \text{ metres} \]

Thus, complete vertical mixing distance is located at 229 metres from the point of discharge. This is the point when the concentration of the bed becomes 90% of the surface concentration and this can be determined by method of images [11].

### 4.3.1 Determination of mass concentration distribution in 2-D mixing zone

Table 3 shows the fitting of the model: \( C_x = \frac{1}{\sqrt{x}} \) into our effluent sample data. The concentration distribution along the course of the river from 1 metre up to 2000 metres is displayed in Table 3. It is evident from the table that the dispersion in the 2-D zone occurs more gradually in the transverse mixing zone i.e far-field. Figure 5 is demonstrative. It shows the steepness of dispersion at near field and the gradual dispersion at far field even beyond 2km.

We employ the relation

\[ C_x = \frac{1}{\sqrt{x}} \]

\[ C_{5(PO_4)} = [0.868 \times \frac{1}{\sqrt{5}}] = 0.447 \]

\[ C_{1000(PO_4)} = [0.868 \times \frac{1}{\sqrt{1000}}] = 0.028 \]

For instance,

\[ C_{1500(BOD)} = [16.467 \times \frac{1}{\sqrt{1500}}] = 0.428 \]

### 4.4 Determination of Location of Complete Transverse and Longitudinal Mixing

The distance to the location where complete transverse and longitudinal mixing occur \( (L_{nh}) \) is referred to as the far-field mixing and is determined with the following relations

\[ L_{nh} = 0.4 \frac{u_h^2}{E_y} \]  

(6)

From [11],

\[ E_y = \kappa_y U h, \kappa_y = 0.5 \pm 50\% , \text{ so} \]

\[ L_{nh} = 0.4 \times 1.35 \times 30^2 = 46.37 \]

Table 3 exemplifies the decay of each of the pollutant parameters along the river course. Thus about 1 km downstream from point source of pollution, complete transverse and longitudinal mixing seem to have occurred.

**Table 2: Concentration Dispersion in 3-D mixing zone**

<table>
<thead>
<tr>
<th>A(m)</th>
<th>C_x</th>
<th>PO_4</th>
<th>NO_3</th>
<th>Cd</th>
<th>Cu</th>
<th>Fe</th>
<th>Pb</th>
<th>Turb.</th>
<th>Temp.</th>
<th>Feacal</th>
<th>Coliform</th>
<th>pH</th>
<th>DO</th>
<th>BOD</th>
<th>COD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
<td>0.868</td>
<td>0.933</td>
<td>0.77</td>
<td>0.503</td>
<td>0.717</td>
<td>30</td>
<td>27.433</td>
<td>126.33</td>
<td>6.16</td>
<td>0.267</td>
<td>16.467</td>
<td>277.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.434</td>
<td>0.467</td>
<td>0.385</td>
<td>0.255</td>
<td>0.531</td>
<td>15</td>
<td>13.717</td>
<td>63.165</td>
<td>3.00</td>
<td>0.134</td>
<td>8.234</td>
<td>138.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.333</td>
<td>0.289</td>
<td>0.311</td>
<td>0.256</td>
<td>0.167</td>
<td>0.686</td>
<td>9.99</td>
<td>9.135</td>
<td>42.068</td>
<td>0.2051</td>
<td>0.089</td>
<td>5.484</td>
<td>92.45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.25</td>
<td>0.217</td>
<td>0.233</td>
<td>0.193</td>
<td>0.126</td>
<td>0.179</td>
<td>7.5</td>
<td>6.858</td>
<td>31.583</td>
<td>1.54</td>
<td>0.067</td>
<td>4.117</td>
<td>69.41</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.20</td>
<td>0.174</td>
<td>0.187</td>
<td>0.151</td>
<td>0.101</td>
<td>0.112</td>
<td>6.0</td>
<td>5.487</td>
<td>25.266</td>
<td>1.232</td>
<td>0.053</td>
<td>3.293</td>
<td>55.53</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.177</td>
<td>0.145</td>
<td>0.156</td>
<td>0.129</td>
<td>0.084</td>
<td>0.085</td>
<td>0.120</td>
<td>5.01</td>
<td>4.581</td>
<td>21.097</td>
<td>1.029</td>
<td>0.045</td>
<td>2.750</td>
<td>46.37</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.143</td>
<td>0.124</td>
<td>0.133</td>
<td>0.110</td>
<td>0.072</td>
<td>0.724</td>
<td>0.103</td>
<td>4.29</td>
<td>3.923</td>
<td>18.065</td>
<td>0.881</td>
<td>0.038</td>
<td>2.355</td>
<td>39.71</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.125</td>
<td>0.109</td>
<td>0.117</td>
<td>0.096</td>
<td>0.063</td>
<td>0.633</td>
<td>0.090</td>
<td>3.75</td>
<td>3.429</td>
<td>15.791</td>
<td>0.77</td>
<td>0.033</td>
<td>2.058</td>
<td>34.71</td>
<td></td>
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<tr>
<td>9</td>
<td>0.111</td>
<td>0.096</td>
<td>0.104</td>
<td>0.085</td>
<td>0.056</td>
<td>0.562</td>
<td>0.080</td>
<td>3.33</td>
<td>3.045</td>
<td>14.023</td>
<td>0.684</td>
<td>0.030</td>
<td>1.828</td>
<td>30.85</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>0.087</td>
<td>0.093</td>
<td>0.077</td>
<td>0.050</td>
<td>0.506</td>
<td>0.072</td>
<td>3.0</td>
<td>2.743</td>
<td>12.633</td>
<td>0.616</td>
<td>0.027</td>
<td>1.647</td>
<td>27.76</td>
<td></td>
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</tbody>
</table>

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The analytics employed in this study has enabled us to show that complete vertical, lateral and longitudinal mixing occur 1.8km downstream from the point of effluent discharge. This stretch of the river is referred to as the sacrificial stretch. This portion is exposed to concentration values above Environmental Quality Standard (EQS) requirements. The effluent concentration parameters at this distance appears to remain constant and may remain so further downstream beyond 2km judging by the screen plot of Figure 6. Statistically, the figure indicates that the standard deviation of sample concentration with distance remains constant.

From 1.8km downstream further the river course, the vegetation in the river banks generates copious oxygen during the night. In addition, the atmospheric pressure aids adsorption of available oxygen in the river thereby minimizing the effects of oxygen sag that has been established in the first 1.8km of oxygen depletion. At the same time, the murkiness of the river starts to get clearer due to gradual settlement of particles at the river bed. Other pollutant parameters also start getting reduced in their effects. And after a considerable distance downstream, assimilative capacity of the river will start bouncing back if no further effluent loading is experienced. And this is the case with Ikpoba River before it joins the estuary that leads to Atlantic Ocean.

The objectives of this study as earlier stated were fully realized. The model which describes the dispersion mechanism was developed. Moreover, our results show that at about 1800 metres, complete transverse and longitudinal mixing was attained and effluents' concentrations became statistically insignificant in value.

REFERENCES


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