A COMPARATIVE ANALYSIS OF THE EFFECTIVE ELASTIC CONSTANTS OF LAMINATED COMPOSITE PLATES OBTAINED BY VARIOUS CLOSED-FORM AND SERIES SOLUTIONS

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ABSTRACT:
Results of various finite element and closed form models developed in the attempt to evaluate and establish accurate values of the Young’s modulus, \( E \); the shear modulus \( G \); and the Poisson’s ratio, \( \nu \) for laminated composite plates having soft matrix and high fibre volume fraction are discussed in this paper. Their merits and limitations highlighted. The Finite Element Energy Method (FEEM) as a tool for the prediction of effective elastic constants for Flexible Matrix composites is proffered here as a model having no better alternative yet.

Keywords: Flexible-matrix-composites Effective-elastic-constants; Finite-element-energy-method

INTRODUCTION:
There are several closed form solutions that exist for the determination of effective elastic constants of a uniaxial composite layer. The works of Hashin, chamis and sendecky, [1,2] discuss several such solutions. However, it has been discovered that different methods give different results for some of the elastic constants. Apart from the two elastic constants, \( E_L \) and \( \nu_{TL} \) which seem to agree for all methods, experimental results grossly indicate that the rest of the elastic constants hardly ever agree as obtained from any one method with another, hence the need to develop a reliable closed form solution for this purpose [3].

Table 1: Effective elastic constants by different methods

<table>
<thead>
<tr>
<th>Method</th>
<th>( E_L ) (N/mm(^2))</th>
<th>( E_T ) (N/mm(^2))</th>
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<th>( G_{LT} ) (N/mm(^2))</th>
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<td>Rule of mixtures</td>
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<td>Whitney and</td>
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<td>1021.0</td>
<td>0.39668</td>
<td>0.87210*</td>
<td>69.20</td>
<td>----</td>
</tr>
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<td>Riley</td>
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<td>0.92016*</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes: [1]* Assumed \( V_{TT} \) For which \( E_T \) are given
Symbols/Notations:

\( C \) = elastic stiffness of the composite
\( e \) = modified strains
\( E_f \) = Young’s modulus of the fibre material
\( E_m \) = Young’s modulus of the matrix material
\( E_L \) and \( E_T \) = Longitudinal and transverse Young’s modulus of the composite
\( G_{LT} \) and \( G_{TT} \) = Longitudinal and transverse shear moduli of the composite
\( U \) = Strain energy of the model
\( u \) = applied displacements
\( V_f \) = fibre volume fraction
\( V_{TL} \) and \( V_{TT} \) = Longitudinal and transverse Poisson’s ratios
\( v_f \) = fibre Poisson’s ratio
\( v_m \) = matrix Poisson’s ratio
\( \varepsilon \) = strains
\( \varepsilon_{\text{engineering}} \) = engineering shear strains.

The tendency for disparities to appear in the results of these methods become even more prominent in composites consisting of very stiff fibre with very soft matrix such as urethane. Here, the fibre volume fraction is usually very high. Such a composite material is usually employed in the manufacture of bearingless rotor systems and drive shafts [4,5].

Table 1 shows the results of elastic constants obtained from some existing closed form solutions working with flexible matrix composites.

### Table 2: Effective elastic constants by different methods

<table>
<thead>
<tr>
<th>Method</th>
<th>( E_L ) N/mm(^2) (x10(^4))</th>
<th>( E_T ) N/mm(^2)</th>
<th>( V_{TL} )</th>
<th>( V_{TT} )</th>
<th>( G_{LT} ) N/mm(^2)</th>
<th>( G_{TT} ) N/mm(^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rule of mixtures</td>
<td>15.31</td>
<td>29.18</td>
<td>0.3518</td>
<td>----</td>
<td>9.73</td>
<td>----</td>
</tr>
<tr>
<td>Halpin-Tsai</td>
<td>15.31</td>
<td>72.36</td>
<td>0.3518</td>
<td>----</td>
<td>16.97</td>
<td>----</td>
</tr>
<tr>
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<td>15.31</td>
<td>72.78</td>
<td>0.3518</td>
<td>----</td>
<td>43.74</td>
<td>----</td>
</tr>
<tr>
<td>Greszezuk</td>
<td>15.31</td>
<td>227.40</td>
<td>0.3518</td>
<td>----</td>
<td>6623.0</td>
<td>----</td>
</tr>
<tr>
<td>Whitney and Riley</td>
<td>15.31</td>
<td>1518.0</td>
<td>0.3517</td>
<td>0.8580*</td>
<td>16.97</td>
<td>----</td>
</tr>
<tr>
<td>Foye</td>
<td>15.31</td>
<td>1725.0</td>
<td>0.3518</td>
<td>0.4811</td>
<td>38.98</td>
<td>----</td>
</tr>
<tr>
<td>Hahn</td>
<td>15.31</td>
<td>67.41</td>
<td>0.3518</td>
<td>----</td>
<td>16.94</td>
<td>16.90</td>
</tr>
<tr>
<td>Hashin (LB)</td>
<td>15.31</td>
<td>1599.0</td>
<td>0.3516</td>
<td>0.8577</td>
<td>16.94</td>
<td>16.90</td>
</tr>
<tr>
<td>Hashin (UB)</td>
<td>15.31</td>
<td>1599.0</td>
<td>0.3517</td>
<td>0.9995</td>
<td>46700.0</td>
<td>391.50</td>
</tr>
</tbody>
</table>

Notes: {1}* Assumed \( V_{TT} \) For which \( E_T \) are given
{2} LB = Lower Bound, UB = Upper Bound

We note here, as we stated before, that all the methods give the same values for \( E_L \) and \( V_{TL} \), indicating that even the Rule of Mixtures is quite appropriate for these properties. However, on the contrary, the shear moduli, \( G_{LT} \) and the transverse properties \( G_{TT} \) differ quite significantly.

For a composite made of much softer matrix and much higher fibre volume fraction, comparative results (see table 2 please) show that the differences in the values obtained by the different methods are much higher than those given in table 1. We also note the large difference between the Hashin lower and upper bounds of \( G_{LT} \), \( G_{TT} \) and \( E_T \).

In their various methods, Halpin Tsai [6,7] used some empirical factors for \( E_T \) and \( G_{LT} \) just as did Ekvall [8,9]. Greszezuk [10] modified the matrix moduli by assuming that the matrix is under plane
stress in one of the transverse directions for its elastic modulus and that it is restrained in both the transverse directions for its shear modulus. Riley and Whitney [11] assumed values for \( V_{TT} \) in their work to obtain \( E_T \) whereas Hashin [12] developed equations for these constants by employing minimum potential energy, giving upper bounds and minimum complimentary energy giving low bounds on a model consisting of a cylindrical fibre enclosed by a cylindrical matrix [fig1]. He proposed that for stiff fibre the upper bounds of \( E_T \) and \( G_{TT} \) can be used but for fibre much stiffer than the matrix [13], the lower bounds should be used. In his investigations, Hahn [14] employed Hashins lower bounds to obtain his values whereas Foye [15] used, instead, a finite element method and obtained stresses for applied strains, which, in turn were used with the other orthotropic stress-strain relations to determine the elastic constants. Next, he compared his results with the existing solutions and selected Whitney and Riley solutions for \( E_T \). The big limitation in his approach is that the stress distribution on a finite element model would be found non-uniform for composites made of very stiff fibres and very soft matrix if they possess very high fibre volume fraction.

So far, we see the great difficulty involved in trying to select one single method to predict the properties of soft matrix composites.

**Purpose of the work:**
The purpose of this work is to obtain by the interaction of the works of Hashin and Rosen; Iwona, Lee and Middya et al [16-20] and also those of Gupta [21], a single model that would effectively predict, to a very reasonable extent, all the elastic constants in a composite with very soft matrix. This model is found in the Finite Element Energy Method (FEEM).

**THE FINITE ELEMENT ENERGY METHOD (FEEM)**
Hashin and Rosen in the development of their equations employed a cylindrical fibre enclosed by a cylindrical matrix (see fig 1) and by applying the minimum potential and minimum complimentary energy approaches arrived at the values of the elastic constants found in tables 1 and 2. Gupta, improving on the works of Hashin and Rosen suggested the introduction of one Representative Volume Element (RVE) for this analysis. He applied to it boundary conditions subjected to controlled displacements (see fig 2).
By employing Nine Representative Volume Element, instead, (Fig 3) as a cross-check to the results of one Representative Volume Element given, the strain energy of the model is obtained as follows:

\[
U = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl},
\]

(1)

with boundary conditions:

\[
U_i = \varepsilon_{ij} X_j
\]

(2)

Equation (2) produces the following strain relations:

\[
\varepsilon_{11} = \bar{\varepsilon}_1 / X_1, \quad \varepsilon_{22} = \bar{\varepsilon}_2 / X_2,
\]

and \( \varepsilon_{33} = \bar{\varepsilon}_3 / X_3 \)

(3)

when only normal strains are expected to exist and,

\[
Y_{12} = \bar{\varepsilon}_1 / X_2, \quad \bar{\varepsilon}_2 = 0, \text{ etc}
\]

(4)

when only one shear strain is to exist.

**STRAIN-STIFFNESS-ENERGY RELATIONS:**

Equation (1) can be re-written in a more convenient form:

\[
U = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl}
\]

(5)

Here \( \varepsilon_1, \varepsilon_2, \) and \( \varepsilon_3 \) are normal strains and \( \varepsilon_4, \varepsilon_5, \) and \( \varepsilon_6 \) are shear strains.

**Normal Stiffness:** For all shear strains equal to zero, we get by expanding equation (5),

\[
U = \frac{1}{2} C_{11} \varepsilon_1^2 + C_{22} \varepsilon_2^2 + C_{33} \varepsilon_3^2 + 2C_{12} \varepsilon_1 \varepsilon_2 + 2C_{13} \varepsilon_1 \varepsilon_3 + 2C_{23} \varepsilon_2 \varepsilon_3
\]

(6)

Let us further simplify the notations by denoting,

\[
\frac{1}{2} C_{11} = C_1, \quad \frac{1}{2} C_{22} = C_2, \quad \frac{1}{2} C_{33} = C_3,
\]

\( C_{12} = C_4, \quad C_{13} = C_5, \quad C_{23} = C_6, \)

and

\[
\varepsilon_1 = \varepsilon_1, \quad \varepsilon_2 = \varepsilon_2, \quad \varepsilon_3 = \varepsilon_3, \quad \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 0,
\]

(7)

Equation (6) then becomes,

\[
U = C_i \varepsilon_i, \quad i = 1, 2, \ldots, 6
\]

(8)

Here \( \varepsilon_i \) are applied modified strains, \( U \) is the resulting strain energy, and \( C_i \) are to be determined.

If we denote \( \varepsilon_{pi} = \) Prescribed modified strains for the displacement boundary condition, \( p, \) where \( p = 1, 2, \ldots, 6, \) and \( U_p = \) resulting strain energy due to \( \varepsilon_{pi}, \) equation (8) then becomes:

\[
U_p = \varepsilon_{pi} C_i
\]

(9)

The following six sets of displacements boundary conditions to produce desired normal strains and energies are prescribed

\( \varepsilon_1 = \varepsilon_1, \quad \varepsilon_2 = \varepsilon_2, \quad \varepsilon_3 = \varepsilon_3, \quad \varepsilon_4 = \varepsilon_4, \quad \varepsilon_5 = \varepsilon_5, \quad \varepsilon_6 = \varepsilon_6 \):

\( p = 1: \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0, \) giving \( \varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33}, \)

and all other \( \varepsilon_{ii} = 0, \)

\( p = 2: \varepsilon_2 = \varepsilon_3 = \varepsilon_1 = 0, \) giving \( \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{11}, \)
and all other $e_{3i} = 0$.

$p = 3$: $e_3 = e_x e_1 = e_2 = 0$, giving $e_{33} = e_{22}$
and all other $e_{3i} = 0$.

$p = 4$: $e_1 = e_x e_2 = e_y$, $e_3 = 0$, giving $e_{41} = e_{2x}$, $e_{42} = e_{2y}$, $e_{44} = e_{xy}$, and all other $e_{4i} = 0$.

$p = 5$: $e_1 = e_x$, $e_2 = 0$, $e_3 = e_z$, giving $e_{51} = e_{2x}$, $e_{53} = e_{2z}$, $e_{55} = e_{xz}$, and all other $e_{5i} = 0$, and

$p = 6$: $e_1 = 0$, $e_2 = e_y$, $e_3 = e_z$, giving $e_{62} = e_{2y}$, $e_{63} = e_{2z}$, $e_{66} = e_{xy} e_{xz}$, and all other $e_{6i} = 0$.

When these strains are substituted, equation (9) becomes:

$$\begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4 \\
U_5 \\
U_6
\end{bmatrix} =
\begin{bmatrix}
e_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & e_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & e_{33} & 0 & 0 & 0 \\
e_{41} & e_{42} & 0 & e_{44} & 0 & 0 \\
e_{51} & 0 & e_{53} & 0 & e_{55} & 0 \\
0 & e_{62} & e_{63} & 0 & 0 & e_{66}
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4 \\
C_5 \\
C_6
\end{bmatrix}$$

(10)

The normal stiffness vector can then be determined by $\{C\} = \{e\}^{-1} \{U\}$ (11)

By using the relations giving by equation (7) the normal stiffness matrix can then be determined. Shear stiffness.

For $e_1 = e_2 = e_3 = 0$, equation (5)

$$U = (C_{44} e_{42} + C_{55} e_{52} + C_{66} e_6)$$

Where $e_4$, $e_5$, and $e_6$ are engineering shear strains, $\gamma_1$, $\gamma_2$, and $\gamma_3$ (or $\gamma_{xy}$, $\gamma_{xz}$, $\gamma_{yz}$), respectively.

The following three sets of displacement boundary conditions are prescribed to produce these strains separately:

$p = 7$: $e_4 = \gamma_{xy}$, $e_5 = e_6 = 0$, giving, $C_{44} = 2U_7 / \gamma_{xy}$.

$p = 8$: $e_5 = \gamma_{xz}$, $e_4 = e_6 = 0$, giving, $C_{55} = 2U_8 / \gamma_{xy}$ and

$p = 9$: $e_6 = \gamma_{yz}$, $e_4 = e_5 = 0$, giving,

$$C_{66} = 2U_9 / \gamma_{yz}.$$  \{13\}

By using the matrix $[C]$ thus determined can be used to compute the effective elastic constants of the composite.

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**Table 3**: Applied displacements for different boundary conditions
(2D = plane strain two dimensional, 3D = three dimensional)

<table>
<thead>
<tr>
<th>Load (P)</th>
<th>Analysis</th>
<th>Applied displacements on sides (surfaces)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cond. (P)</td>
<td>type</td>
<td>Load (ADHE)</td>
</tr>
<tr>
<td>1</td>
<td>2D</td>
<td>$U_i=0$</td>
</tr>
<tr>
<td>2</td>
<td>2D</td>
<td>$U_i=0$</td>
</tr>
<tr>
<td>3</td>
<td>3D</td>
<td>$U_i=0$</td>
</tr>
<tr>
<td>4</td>
<td>2D</td>
<td>$U_i=0$</td>
</tr>
<tr>
<td>5</td>
<td>3D</td>
<td>$U_i=0$</td>
</tr>
<tr>
<td>6</td>
<td>3D</td>
<td>$U_i=0$</td>
</tr>
<tr>
<td>7(A)</td>
<td>2D</td>
<td>Linear</td>
</tr>
<tr>
<td>(B)</td>
<td>2D</td>
<td>$U_i=0$</td>
</tr>
<tr>
<td>(C)</td>
<td>2D</td>
<td>---</td>
</tr>
<tr>
<td>8(A)</td>
<td>3D</td>
<td>$U_i=0$</td>
</tr>
<tr>
<td>(B)</td>
<td>3D</td>
<td>$U_i=0$</td>
</tr>
<tr>
<td>9(A)</td>
<td>3D</td>
<td>$U_i=0$</td>
</tr>
<tr>
<td>(B)</td>
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<td>$U_i=0$</td>
</tr>
</tbody>
</table>
Table 4: Comparison of effective elastic constants by different methods

\[ E_f = 2.075 \times 10^5 \text{ N/mm}^2, \quad E_m = 68.98\text{ N/mm}^2, \quad \nu_f = 0.3, \quad \nu_m = 0.49444, \quad V_f = 0.5 \]

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<tr>
<th>Method</th>
<th>( E_L ) N/mm(^2) (x10(^4))</th>
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<td>Hashin (UB)</td>
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<td>1093.0</td>
<td>0.39668</td>
<td>0.99657</td>
<td>26560.0</td>
<td>275.3</td>
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<td>FEEM (1 RVE)</td>
<td>10.42</td>
<td>643.0</td>
<td>0.39615</td>
<td>0.92009</td>
<td>71.12 (A) 400.0, (B)77.8, (C)73.1</td>
<td></td>
</tr>
<tr>
<td>FEEM (9 RVE)</td>
<td>10.42</td>
<td>644.0</td>
<td>0.39620</td>
<td>0.92016</td>
<td>71.12 (A) 82.4, (B)59.7, (C)47.8</td>
<td></td>
</tr>
</tbody>
</table>

Notes: (1) (A), (B), and (C) represent different boundary conditions described in Table 3  
(2)\(^*\) Assumed \( V_{TT} \) for which \( E_T \) are given  
(3) LB = Lower Bound, UB = Upper Bound

DISCUSSIONS/RESULTS

Table 3 gives the various applied displacements on the sides/surfaces of these models according to the strain-boundary conditions explained earlier. This set of displacements represents nine separate problems to be analyzed, yielding nine independent elastic constants. Where the transverse dimensions along X and Y axes of the model are equal, which could be the case for most general composites, the boundary condition (1) is the same as (2), (5) and (6), and (8) as (9), leaving only six independent boundary conditions. This reduced set of boundary conditions would give six independent elastic constants, \( E_L \),
$E_T$, $V_{TL}$, $V_{TT}$, $G_{LT}$, and $G_{TT}$. Figures 2 and 3 show the two-dimensional finite element grids employed. The three-dimensional grids are exactly the same except that there are three elements in the Z-direction. Six separate finite element analyses were performed according to the boundary conditions given in table 3 which give six values of strain energy for the six sets of applied strains.

The six elastic constants were computed by using equations (11) and (13) and the other relations. These results are set into tables 4 and 5 to compare with results of the other methods as formerly obtained in tables 1 and 2 (see and compare for yourself). In tables 1 and 4, a steel-urethane composite was employed, where Young’s modulus for urethane is 68.98 N/mm$^2$ and fibre volume fractions is 50 percent and in tables 2 and 5, a steel-urethane composite having Young’s modulus for urethane, 7.6 N/mm$^2$ and the fibre volume fraction, 74 percent.

CONCLUSIONS:

It is worthy of note that all elastic constants by the FEEM for one RVE and nine RVEs are found the same except for the values of $G_{TT}$ [see tables 4 and 5 to confirm]. This actually goes to prove the effectiveness of the work of Gupta. The little disparity in the values of $G_{TT}$ can easily be understood if we recognize that more energy is required for the condition at (A), in the case of one RVE, than in nine RVE where the energy is reduced drastically because the comparatively rigid fibres are allowed to rotate freely under the application of the classical boundary conditions. Since the resulting values should not be less accurate, better results would be given by more RVE’s in the model, with the boundary conditions at (A). In conclusion, therefore, the FEEM can be regarded as the all-in-one-tool so far available for the effective prediction of elastic constants in composites possessing very stiff fibres and very soft matrix.

REFERENCES

Springs, California, 1966, pp. 205 – 263.


